Here, we continue the problem from last time.

Recall $C_\varepsilon(x) = \{y \in E : d(x,y) \leq \varepsilon\}$ is a closed ball of radius ε centered at x

Def: E is totally bounded if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ and $x_1, \dots, x_n$ such that $E = \bigcup_{i=1}^{n} C_\varepsilon(x_i)$

Note: $S \subset \mathbb{R}^n$, S is bounded $\Rightarrow$ S is totally bounded.
where $n \in \mathbb{N}$ is finite.

<u>Example</u>: $E = \{f : [0,1] \mapsto [0,1]\}$.

$$d(f,g) = l.u.b\{|f(x) - g(x)| : x \in [0,1]\}$$

Q: Is $d(f,g)$ a distance?

1) $d(f,g) \geq 0$ $\quad$ V

2) $d(f,g) = 0 \Leftrightarrow f = g$ $\quad$ V

3) $d(f,g) = |f(x) - g(x)| = |g(x) - f(x)| = d(g,f)$

4) $f, g, h \in E$.

$$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)|$$

$$\leq |f(x)-g(x)| + |g(x)-h(x)|$$
$$\leq d(f,g) + d(g,h)$$

then $d(f,g)+d(g,h)$ is an upper bound of set $\{|f(x)-h(x)| : x \in [0,1]\}$.

since $d(f,h)$ is the l.u.b of this set,

$$d(f,h) \leq d(f,g) + d(g,h).$$

Thus, $d(f,g)$ is a distance.

Claim E is bounded. Indeed, $f_0(x)=0$

$E \subset B_2(f_0)$ since $|f(x)-f_0(x)| = |f(x)| \leq 1$

$\forall f(x) \in E$, then $d(f,f_0) \leq 1 < 2$.

Suppose we have $f_n \in E$, $n \in \mathbb{N}$, such that

$d(f_i, f_j) = 1$, $\forall i \neq j$.

Assume $E \subset \bigcup_{k=1}^{r} C_{1/3}(g_k)$, if f_{n_1} and $f_{n_2} \in$

$C_{1/3}(g_{k_0}) \Rightarrow d(f_{n_1}, f_{n_2}) \leq d(f_{n_1}, g_{k_0}) + d(g_{k_0}, f_{n_2})$

$\leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. This contradicts with $d(f_i, f_j) = 1$.

Now, suppose $f_n : [0,1] \mapsto [0,1]$, $d(f_k, f_j) = 1$ if $k \neq j$.

$$f_n(x) = \begin{cases} 0, & \text{if } x \neq \frac{1}{n} \\ 1, & \text{if } x = \frac{1}{n} \end{cases}$$

so $d(f_i, f_j) = 1$, $\forall i \neq j$.

So, E is bounded but not totally bounded.

<u>Exercise</u>: E is complete and totally bounded, then E is compact.

~~proof~~ By contradiction, let assume E is totally bounded and complete, but not compact.

Let $E = \bigcup_{i \in I} U_i$, since E is not compat.

$E \neq \bigcup_{k=1}^{n} U_{i_k}$ for any n and $i_1, i_2, \cdots, i_n$.

Since E is totally bounded, $\exists\ x_1^{(r)}, \cdots, x_{n_r}^{(r)}$ s.t. $E = \bigcup_{j=1}^{n_r} C_{1/r}(x_j^{(r)})$.

When $r=1$, $\exists\ i_1$ s.t. $A_1 = C_1(x_{i_1}^{(1)})$ is not included in a finite union of U_i

when $r=2$, $A_1 \subset E = \bigcup_{j=1}^{n_2} C_{1/2}(x_j^{(2)})$

$$\Rightarrow A_1 = \bigcup_{j=1}^{n_2} \left(C_{1/2}(x_j^{(2)}) \cap A_1 \right)$$

Then, $\exists\ i_2$, s.t. $A_2 = C_{1/2}(x_{i_2}^{(2)}) \cap A_1$ is not included in a finite union of U_i

I am constructing A_n such that $A_{n+1} \subset A_n$ and each A_n is closed. If $x, y \in A_n \subset C_{1/n}(x_{i_n}^{(n)})$, $d(x,y) \leq 2 \cdot \frac{1}{n} = \frac{2}{n}$

Let $x_n \in A_n$, $r \geq s \Rightarrow x_r \in A_r \subset A_s$

$x_s \in A_s$

$\Rightarrow d(x_r, x_s) \leq \frac{2}{s} \Rightarrow x_n$ is Cauchy.

Since E is complete, $x_n \to x \in E$.

So, $\exists\, i_0 \in I$, s.t. $x \in U_{i_0}$

Since U_{i_0} is open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U_{i_0}$

Select n s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ and $\frac{2}{n} < \frac{\varepsilon}{2}$

If $y \in A_n$ then $d(y, x) \leq d(y, x_n) + d(x_n, x)$

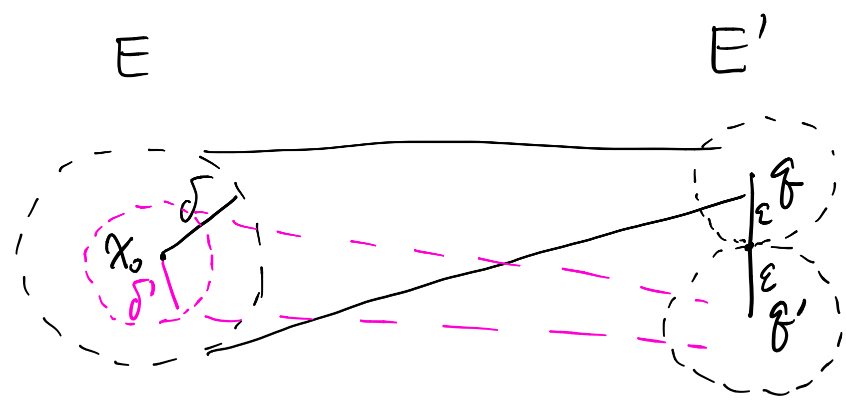
$< \frac{2}{n} + \frac{\varepsilon}{2} < \varepsilon$. Thus, $A_n \subset U_{i_0}$

This contradicts with A_n cannot be covered by finite union of U_i.

<u>Limits</u>

<u>Obs</u>: If x_0 is a cluster point of E and $f: E \mapsto E'$ (or $E - \{x_0\} \mapsto E'$), then if $\lim_{x \to x_0} f(x)$ exits. it is unique.

<u>proof</u> Assume $q = \lim_{x \to x_0} f(x)$ and $q' = \lim_{x \to x_0} f(x)$.



$\varepsilon = \frac{d(q,q')}{2} > 0$

If $q \neq q'$, let $\varepsilon = \frac{d(q,q')}{2}$, $\exists\, \delta > 0, \delta' > 0$ such that

$$\begin{cases} d(x,x_0) < \delta \Rightarrow d(f(x), q) < \varepsilon \\ d(x,x_0) < \delta' \Rightarrow d(f(x), q') < \varepsilon \end{cases}$$

If $\alpha = \min\{\delta, \delta'\}$, $B_\alpha(x_0) = \{x_0\}$

$\Rightarrow x_0$ is not a cluster point of E

This contradicts with the assumption.

<u>Notation</u> $S \subset E$, x_0 is cluster point in S.

$$\lim_{\substack{x \to x_0 \\ x \in S}} f(x)$$

<u>Example</u> $\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a \\ \{x > a\}}} f(x)$, $f: \mathbb{R} \mapsto \mathbb{R}$.

<u>Proposition</u> $f : E \mapsto E'$, $x_0 \in E$

f is continuous at $x_0 \iff \forall x_n$ s.t. $x_n \to x_0$ $\Rightarrow f(x_n) \to f(x_0)$.